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Correlation functions in a $c = 1$ boundary conformal field theory

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Abstract: We obtain exact results for correlation functions of primary operators in the two-dimensional conformal field theory of a scalar field interacting with a critical periodic boundary potential. Amplitudes involving arbitrary bulk discrete primary fields are given in terms of SU(2) rotation coefficients while boundary amplitudes involving discrete boundary fields are independent of the boundary interaction. Mixed amplitudes involving both bulk and boundary discrete fields can also be obtained explicitly. Two- and three-point boundary amplitudes involving fields at generic momentum are determined, up to multiplicative constants, by the band spectrum in the open-string sector of the theory.

Keywords: Conformal Field Models in String Theory, Tachyon Condensation, Quantum Dissipative Systems.

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Contents

1. Introduction

We consider the problem of computing correlation functions of primary operators in a relatively simple, yet non-trivial, two-dimensional boundary conformal field theory. The system we study involves a scalar field that is free in the two-dimensional bulk but has a periodic boundary interaction, with the period of the interaction potential chosen such that the boundary theory preserves half of the conformal symmetry of the bulk theory.

In the Caldeira-Leggett [1] approach to dissipative quantum mechanics this model describes critical behaviour of a particle subject to dissipation while moving in a periodic potential [2, 3]. The model also arises in connection with quantum Hall edge states [4] and in the study of bosonic string theory in an on-shell open string tachyon background [5]. More recently, a closely related $c = 1$ boundary conformal field theory involving a scalar field with a 'wrong-sign' kinetic term in the bulk and a critical boundary potential has been applied to describe the rolling tachyon field of an S-brane [6]–[9].

For all these applications it is potentially useful to supplement existing results on exact boundary states and partition functions by computations of correlation functions of the primary operators in the theory. This task is aided by the SU(2) current algebra that resides in the model [10]–[12]. For a compact boson, taking values on a circle at the self-dual radius, the SU(2) symmetry allows us to obtain explicitly correlation functions, that involve any bulk primary fields and/or boundary primary fields that do not change the boundary conditions. At other allowed radii we have exact results for only a subset of non-vanishing amplitudes.

We obtain exact scaling dimensions of all primary operators in the boundary theory and this, along with the conformal symmetry, is sufficient to determine low-order correlation functions up to constant coefficients. In particular, we extend the known band structure of the system to include open string states where the string ends couple to different potential strength. The energy eigenvalues of such states enter into correlation functions of boundary condition changing operators. A subset of the results described in this paper were briefly presented in [13].

The paper is organized as follows. In the remainder of this section we define the model and establish our notation. We briefly discuss some related models that have been studied in the recent literature. In section 2 we compute scattering amplitudes involving bulk operators. We identify the bulk discrete primary fields, which play a key role in what follows, and then extend the prescription for computing bulk amplitudes of elementary string excitations, given in [11], to include bulk discrete fields. We explicitly work out one- and two-point functions of bulk discrete fields and verify that our prescription preserves crossing symmetry in the interacting theory. In section 3 we turn our attention to boundary fields, including discrete boundary primary fields. We argue that correlation functions that involve any number of discrete boundary fields, and no other fields, are actually independent of the boundary coupling. Section 4 briefly reviews the free fermion calculation of the open string spectrum [12] and then extends it to include open strings whose ends interact with boundary potentials of different strength. The corresponding calculation in the closed string sector is carried out in an appendix. In section 5 we discuss the problem of more general boundary amplitudes in models where the free boson is non-compact, or compactified at some multiple of the self-dual radius. These involve fields carrying momenta that differ from those of the discrete fields, and possibly also boundary condition changing fields. The prescription we use for discrete fields does not carry over to the general case but conformal symmetry and the open string spectrum from section 4 determine low-order boundary correlators up to multiplicative factors, which may depend on the strength of the boundary coupling. We also give a prescription for computing mixed amplitudes involving both bulk and boundary discrete fields. Section 6 contains a discussion and some directions for future work.

1.1 The system

Our scalar field $\Phi(z,\bar{z})$ is defined on the upper-half-plane, $\mathrm{Im}\, z > 0$, with dynamics governed by the action[1]

$$S = \frac{1}{4\pi}\int d^2z\, \partial\Phi\bar{\partial}\Phi - \frac{1}{2}\int d\tau \left[g e^{i\Phi(\tau)/\sqrt{2}} + \bar{g} e^{-i\Phi(\tau)/\sqrt{2}} \right]. \tag{1.1}$$

The strength and the phase of the periodic boundary interaction is determined by the complex parameter g. For a field that takes value on a finite circle of radius R the boundary action in (1.1) is single valued only if $2\pi R$ is an integer multiple of the period of the potential. This in turn means that the radius R has to be an integer multiple of the self-dual radius of a free boson, $R_{\text{self-dual}} = \sqrt{2}$. The theory at the self-dual radius itself is the simplest and our methods yield exact results for correlation functions of arbitrary primary fields (apart from boundary condition changing fields) in that case. We also consider the other allowed radii, including the the $R \to \infty$ limit of a non-compact scalar, but our results are less complete in those cases.

The period of the potential in (1.1) has been chosen such that the interaction has dimension one under boundary scaling. At this critical period the model is an exact boundary conformal field theory, i.e. half the conformal symmetry of the bulk system is preserved. In the process of establishing the conformal symmetry, the model was also found to have an underlying SU(2) symmetry, which leads to some remarkably simple exact results [10]–[12]. For example, by a judicious choice of regularization, the boundary state of the interacting theory was shown to be given by a global SU(2) rotation, parametrized by g and $\bar{g}$, of the free boundary state and scattering amplitudes involving elementary string excitations were explicitly computed [11]. The open string spectrum exhibits an interesting band structure, first found in a free fermion representation of the model [12]. The bands are analogous to energy bands in condensed matter physics although the conformal symmetry of the string system leads to a spectrum that differs qualitatively from the standard non-relativistic case.

1.2 Related models

Recently there has been renewed interest in $c = 1$ boundary conformal field theory and a number of models, that are related the one studied here, have been considered in the literature.

[1] We have set $\alpha' = 2$.

By formally analytically continuing the field, $\Phi \to -i\Phi$, and taking $g \in \mathbb{R}$ in (1.1) one obtains the following action,

$$S = -\frac{1}{4\pi}\int d^2z\,\partial\Phi\bar{\partial}\Phi - \frac{g}{2}\int d\tau\,\cosh\frac{\Phi(\tau)}{\sqrt{2}}\,, \tag{1.2}$$

which has been applied to describe the rolling tachyon field of an S-brane [6]–[9]. The S-brane instability is signalled by the 'wrong sign' kinetic term in (1.2). Bulk and boundary correlation functions in this unstable theory, that describe closed and open string production in an S-brane background [7, 8] are closely related to correlation functions in the static background defined by (1.1).

Another model studied in recent work [14, 15] has a complex valued boundary potential involving only a single exponential,

$$S = \frac{1}{4\pi}\int d^2z\,\partial\Phi\bar{\partial}\Phi - \frac{\lambda}{2}\int d\tau\,e^{-i\Phi(\tau)/\sqrt{2}}\,. \tag{1.3}$$

Here the boundary interaction is non-hermitean and the connection to (1.1) is more tenuous than in the case of (1.2). Despite being non-hermitean, this model is in many respects simpler than the one defined by (1.1). In either case we can view a perturbative expansion in the boundary coupling, g or λ respectively, in terms of a one-dimensional Coulomb gas. The real valued potential in (1.1) corresponds to charges of both signs, while only same-sign charges occur in the theory with the single exponential, and this severely restricts how the interactions contribute to physical quantities including correlation functions. This is for example reflected in the spectrum of scaling dimensions of boundary operators, which is independent of the coupling λ of the single exponential [15] (see also section 4.3 below), but develops non-trivial dependence on g when we work with the real valued potential in (1.1).

The primary motivation for the study of the action (1.3) is its relation, by analytic continuation in the field, to a theory that describes the decay of an unstable D-brane in string theory,

$$S = -\frac{1}{4\pi}\int d^2z\,\partial\Phi\bar{\partial}\Phi - \frac{\lambda}{2}\int d\tau\,e^{\Phi(\tau)/\sqrt{2}}\,. \tag{1.4}$$

Here the Φ field is interpreted as the embedding time in a worldsheet description of the D-brane and this model is also referred to as timelike boundary Liouville theory [8, 16, 17]. In parallel with the theories with periodic boundary interactions, the study of (1.4) is more manageable than that of (1.2) because here the non-linear boundary term goes to zero at early embedding time, $\Phi \to -\infty$.

1.3 Left-moving SU(2) current algebra

Let us return to the study of (1.1). In the absence of the boundary interaction, the field $\Phi(z,\bar{z})$ satisfies a Neumann boundary condition at $\mathrm{Im}\,z = 0$, and this is conveniently dealt with using the so-called doubling trick or method of images (see f. ex. [18, 19]). The theory on the upper half-plane is then mapped into a chiral theory on the full complex plane and the boundary eliminated. To see how this works we note that away from the boundary the free field can be written as a sum of left- and right-moving (holomorphic and

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anti-holomorphic) components

$$\Phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})\,. \tag{1.5}$$

For $g=0$ the Neumann boundary condition on the real axis $\bar{\phi}(\bar{z}) - \phi(z)|_{z=\bar{z}} = 0$ determines the right-moving field in terms of the left-moving one. We can therefore reflect the right-moving field through the boundary and represent it as a left-moving field in the lower-half-plane. The theory then contains only left-moving fields but a left-moving field at z^* in the unphysical lower-half-plane is to be interpreted as a right-moving field at z in the physical region.[2]

More generally, any quasi-primary field in the bulk separates into a holomorphic and an anti-holomorphic part

$$\Psi_{h,\bar{h}}(z,\bar{z}) = \psi_h(z)\bar{\psi}_{\bar{h}}(\bar{z})\,. \tag{1.6}$$

Using the doubling trick $\bar{\psi}_{\bar{h}}(\bar{z})$ becomes a holomorphic field $\psi_{\bar{h}}(z^*)$, with holomorphic dimension $\bar{h}$. A bulk n-point function $\langle \psi_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\cdots\psi_{h_n,\bar{h}_n}(z_n,\bar{z}_n)\rangle$ in the original theory on the upper-half-plane then becomes a $2n$-point function of holomorphic fields $\langle \psi_{h_1}(z_1)\psi_{\bar{h}_1}(z_1^*)\cdots\psi_{h_n}(z_n)\psi_{\bar{h}_n}(z_n^*)\rangle$ on the infinite plane.

It turns out that the doubling trick can be applied even when the boundary interaction in (1.1) is turned on. This is because the boundary potential can in fact be expressed in terms of left-moving fields only as

$$-\frac{1}{2}\left(g\, e^{i\sqrt{2}\phi(z)} + \bar{g}\, e^{-i\sqrt{2}\phi(z)}\right)\Bigg|_{\mathrm{Im}(z)=0} \tag{1.7}$$

Note that the coefficient in the exponential has changed because on the real axis $\Phi(z,\bar{z}) = 2\phi(z)$.

The operators appearing in the interaction (1.7) are currents of a left-moving SU(2) algebra

$$J_\pm = e^{\pm i\sqrt{2}\phi(z)}, \qquad J_3 = \frac{i}{\sqrt{2}}\partial\phi(z)\,. \tag{1.8}$$

In the boundary action (1.1) the J_+ and J_- currents are integrated along the real axis (the former boundary) and in a perturbative expansion of a bulk correlation function such integrals are repeatedly inserted into the amplitude. As usual, divergences arise when operator insertions coincide but, by a clever choice of regularization, Callan et al. [11] were able to sum the perturbation series explicitly to obtain the exact interacting boundary state. It turned out to be remarkably simple, with the net effect of the interaction being a global SU(2) rotation,

$$U(g_r) = \exp \pi i (g_r J_+ + \bar{g}_r J_-)\,, \tag{1.9}$$

acting on the free Neumann boundary state. The coupling constants g_r and $\bar{g}_r$ that enter in the rotation group element are renormalized from their bare values [11]. In the weak

[2] Our notation follows that of [18]. Both $\bar{z}$ and z^* denote the complex conjugate of z. We use $\bar{z}$ for the argument of a right-moving, anti-holomorphic field $\bar{\phi}(\bar{z})$, but z^* for that of the corresponding left-moving image field $\phi(z^*)$.

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coupling limit g_r approaches g, whereas at strong coupling when $|g| \to \infty$ the renormalized coupling instead goes to a finite value, $|g_r| \to 1/2$. The explicit form of this redefinition has recently been worked out to be [20]

$$g_r = \frac{2g}{\pi|g|} \arctan\left[\tanh\left(\frac{\pi}{2}|g|\right)\right]. \tag{1.10}$$

It is the renormalized coupling that enters in our formulas below and we will drop the r subscript on g_r henceforth.

2. Bulk scattering amplitudes

In this section we consider scattering amplitudes involving bulk fields. General bulk amplitudes are functions of the boundary coupling g and our goal is to determine this dependence. Some bulk amplitudes that are zero in the free theory are nonvanishing in the presence of the boundary interaction. This is because the boundary interaction breaks translation invariance in the target space. It represents a periodic background that can absorb momenta lying in the corresponding reciprocal lattice.

2.1 Bulk primary fields

The bulk theory is that of a free boson. For a non-compact boson there are holomorphic primary fields $e^{ip\phi(z)}$ with conformal weight $h = p^2/2$ for all $p \in \mathbb{R}$ and also the corresponding anti-holomorphic fields. If the boson instead takes value on a circle of finite radius R, then the momentum variable is discrete, $p = n/R$ with $n \in \mathbb{Z}$. Recall that the only radii that are commensurate with the period of the boundary potential are integer multiples of the self-dual radius $R_{\text{self-dual}} = \sqrt{2}$.

At special values of the momentum, $p = \sqrt{2}\, j$, where j is an integer or integer-plus-half, some descendant states have vanishing norm and new primary fields appear, the so-called discrete primaries [21]. We note that these special values of momentum coincide with the allowed momenta at the self-dual radius and are among the allowed momenta at any integer multiple of $R_{\text{self-dual}}$.

The discrete primary fields come in SU(2) multiplets labelled by j and m, with $-j \leq m \leq j$. The discrete fields $\psi_{jm}(z)$ in a given SU(2) multiplet are degenerate in that they all have the same holomorphic conformal weight $h = j^2$. They are composite fields made from certain polynomials in $\partial\phi$, $\partial^2\phi$, etc. accompanied by $e^{i\sqrt{2}m\phi}$.

The ψ_{jm} have the following representation [22],

$$\psi_{jm}(z) \sim \left(\oint \frac{dw}{2\pi i} e^{-i\sqrt{2}\phi(w)}\right)^{j-m} e^{i\sqrt{2}j\phi(z)}, \tag{2.1}$$

where the lowering current is integrated along nested contours surrounding z. The operator products are evaluated using Wick contractions and the free holomorphic propagator

$$\langle\phi(z)\phi(z')\rangle = -\log(z - z'). \tag{2.2}$$

This propagator is also used in normal ordering composite fields, such as those that appear in (2.1), and we refer to this as *bulk normal ordering.*

We adopt the convention

$$\psi_{j,m-1}(z) = c_{j,m} \oint \frac{dw}{2\pi i} e^{-i\sqrt{2}\phi(w)}\, \psi_{jm}(z)\,, \tag{2.3}$$

with $c_{j,m} = [j(j+1) - m(m-1)]^{-1/2}$, for the relative normalization of the ψ_{jm} and the overall normalization is fixed by setting $\psi_{jj} = e^{i\sqrt{2}j\phi}$.[3]

Let us list a few explicit examples for later use. The $j = 0$ case is trivial, $\psi_{0,0} = 1$, and at $j = 1/2$ there is a doublet,

$$\psi_{\frac{1}{2},\frac{1}{2}} = e^{i\phi/\sqrt{2}}\,, \qquad \psi_{\frac{1}{2},-\frac{1}{2}} = e^{-i\phi/\sqrt{2}}\,. \tag{2.4}$$

At $j = 1$ we recognize the SU(2) currents themselves, modulo constant factors,

$$\psi_{1,1} = e^{i\sqrt{2}\phi}\,, \qquad \psi_{1,0} = -i\partial\phi\,, \qquad \psi_{1,-1} = -e^{-i\sqrt{2}\phi}\,, \tag{2.5}$$

and at $j = 3/2$ we find

$$\begin{aligned} \psi_{\frac{3}{2},\frac{3}{2}} &= e^{3i\phi/\sqrt{2}}\,, & \psi_{\frac{3}{2},\frac{1}{2}} &= \frac{-1}{\sqrt{3}}\left((\partial\phi)^2 + \frac{i}{\sqrt{2}}\partial^2\phi\right) e^{i\phi/\sqrt{2}}\,, \\ \psi_{\frac{3}{2},-\frac{3}{2}} &= -e^{-3i\phi/\sqrt{2}}\,, & \psi_{\frac{3}{2},-\frac{1}{2}} &= \frac{1}{\sqrt{3}}\left((\partial\phi)^2 - \frac{i}{\sqrt{2}}\partial^2\phi\right) e^{-i\phi/\sqrt{2}}\,. \end{aligned} \tag{2.6}$$

The operator product expansion (OPE) of holomorphic discrete fields defines a closed operator algebra,

$$\psi_{jm}(z)\psi_{j'm'}(w) = \sum_{J,M} \frac{A^{JM}_{jm,j'm'}\,\psi_{JM}(w)}{(z-w)^{j^2+j'^2-J^2}} + \cdots\,, \tag{2.7}$$

where the $A^{JM}_{jm,jm'}$ are constant OPE coefficients and the ellipsis denotes the contribution from descendants of the ψ_{JM} primaries. The SU(2) symmetry is apparent in the OPE coefficients,

$$A^{JM}_{jm,j'm'} = f(j,j',J)\, C^{JM}_{jm,j'm'} \tag{2.8}$$

where $C^{JM}_{jm,jm'}$ are Clebsch-Gordan coefficients and $f(j,j',J)$ are numbers that can be obtained by direct calculation [22]. The Clebsch-Gordan coefficients restrict the sum over J in (2.7) to $|j-j'| \le J \le j+j'$ and the only value of M that contributes is $M = m+m'$. For $j = 1$ and $j' = 1/2$ we have for example

$$\begin{aligned} \psi_{11}(z)\psi_{\frac{1}{2}\frac{1}{2}}(w) &= (z-w)\ \psi_{\frac{3}{2}\frac{3}{2}}(w) + \cdots \\ \psi_{10}(z)\psi_{\frac{1}{2}\frac{1}{2}}(w) &= \frac{-1/\sqrt{2}}{(z-w)}\,\psi_{\frac{1}{2}\frac{1}{2}}(w) - \sqrt{2}\,\partial\psi_{\frac{1}{2}\frac{1}{2}}(w) + \\ &\quad + (z-w)\left[\sqrt{\frac{2}{3}}\,\psi_{\frac{3}{2}\frac{1}{2}}(w) - \frac{2\sqrt{2}}{3}\,\partial^2\psi_{\frac{1}{2}\frac{1}{2}}(w)\right] + \cdots\,. \end{aligned} \tag{2.9}$$

From the first OPE we see that $f(1,\frac{1}{2},\frac{3}{2}) = 1$ and in the second one the coefficient in front of $\psi_{\frac{3}{2}\frac{1}{2}}$ is $\sqrt{2/3}$ which is the expected Clebsch-Gordan coefficient.

[3]Strictly speaking we need to include a cocycle in the definition of ψ_{jj} in order to ensure locality of the OPE involving ψ_{jm}'s with j an integer plus a half (see for example [19], page 239). Such a cocycle will affect the overall sign of some amplitudes but to avoid clutter we leave it out.

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The bulk theory also contains anti-holomorphic discrete fields, which also form SU(2) multiplets and have their own OPE. A *bulk discrete primary field* is constructed from a pair of holomorphic and anti-holomorphic discrete fields,

$$\Psi_{jm,\bar{j}\bar{m}}(z,\bar{z}) = \psi_{jm}(z)\bar{\psi}_{\bar{j}\bar{m}}(\bar{z})\,. \tag{2.10}$$

The left- and right-moving parts can carry different SU(2) labels subject to the following restrictions.

Let us first consider a compact boson at the self-dual radius. The winding number around the compact circle is given by $w = m - \bar{m}$ and this requires

$$m - \bar{m} \in \mathbb{Z}\,. \tag{2.11}$$

It follows that the difference $j - \bar{j}$ must also be an integer. We note also that the spin $h - \bar{h} = j^2 - \bar{j}^2$ takes an unphysical value unless $j - \bar{j} \in \mathbb{Z}$.

The target space momentum carried by the field (2.10) is $p_{m,\bar{m}} = (m+\bar{m})/\sqrt{2}$. The allowed momentum values at the self-dual radius, $R_{\text{self-dual}} = \sqrt{2}$, correspond precisely to the integer values taken by $m + \bar{m}$. This means that, at the self-dual radius, the bulk primary fields in the theory consist of the bulk discrete primaries, and no others. As a result all bulk correlation functions in the theory at the self-dual radius can be computed by the SU(2) methods discussed below.

If the boson is compactified at some integer multiple of the self-dual radius, $R = q\sqrt{2}$ with $q \in \mathbb{Z}$, the restriction from the winding number becomes more stringent

$$m - \bar{m} = 0 \mod q\,, \tag{2.12}$$

but at the same time there are additional bulk primary fields in the theory that carry momenta $p = n/(q\sqrt{2})$ with $n \in \mathbb{Z}$. The operator product of the SU(2) currents (1.8) with these additional primary fields is non-local.

In the limit of a non-compact boson there can be no winding number, which amounts to the restriction $\bar{m} = m$ on the bulk discrete primary fields. On the other hand, the target space momentum of a non-compact boson is unrestricted leading to a continuum of bulk primary fields

$$\Psi_p(z,\bar{z}) = e^{ip(\phi(z)+\bar{\phi}(\bar{z}))}\,, \tag{2.13}$$

with $p \in \mathbb{R}$.

2.2 Method of rotated images

Callan *et al.* [11] gave a simple prescription for amplitudes that describe the scattering of elementary string excitations off the interacting worldsheet boundary. Let us briefly review this prescription, which we refer to as *the method of rotated images.*

Left-moving string excitations are created and destroyed by $\partial\phi$ so for each incoming field a factor of $\partial\phi(z)$ is inserted in the upper-half-plane. For each right-moving outgoing excitation we normally would insert $\bar{\partial}\bar{\phi}(\bar{z})$ also above the real axis, but using the doubling trick we instead insert the reflected operator $\partial\phi(z^*)$ in the lower-half-plane. This is possible even with the boundary interaction turned on because the anti-holomorphic fields $\bar{\partial}\bar{\phi}$ commute with the holomorphic SU(2) currents in the interaction and reflect through the underlying Neumann boundary condition exactly as in the free theory.

The integration contours of the SU(2) currents that appear in the boundary interaction can now be deformed into the lower half-plane and moved off to infinity. This leaves behind closed integration contours around each image field in the lower half-plane and, since the $\partial\phi(z^*)$ fields are themselves proportional to left-moving SU(2) currents, the current algebra ensures that there will be no cuts generated. The terms in the perturbative series involve an ever higher number of nested contours surrounding each $\partial\phi(z^*)$ field insertion, but these sum up to very simple result: a global SU(2) rotation acting on each inserted field. The global SU(2) element is given by (1.9), which for $g = \bar{g}$ amounts to a rotation by the angle $2\pi g$ about the J_1 axis, giving

$$\partial\phi(z^*) \to \cos(2\pi g)\,\partial\phi(z^*) - \frac{\sin(2\pi g)}{\sqrt{2}}\left(e^{i\sqrt{2}\phi(z^*)} - e^{-i\sqrt{2}\phi(z^*)}\right). \tag{2.14}$$

Finally one evaluates the resulting correlator of left-moving fields using the free holomorphic propagator (2.2).

2.3 Bulk amplitudes involving discrete primary fields

The method of rotated images can be generalized to compute scattering amplitudes involving arbitrary bulk discrete fields. The key to this is the fact that the boundary interaction (1.7) involves left-moving SU(2) currents which act on the discrete fields in a well defined manner. As discussed above, for a compact boson at the self-dual radius, all bulk primary fields are discrete fields so the method can be applied to all bulk amplitudes of interest in that model. At other allowed radii there are non-vanishing amplitudes to which the method cannot be applied. In this case, bulk amplitudes can involve both the discrete bulk fields and fields carrying generic target space momenta. The only constraint from momentum conservation is that the total momentum of all the fields in a given correlator add up to an integer multiple of the lattice momentum of the periodic background boundary potential. The operator product of the currents in the boundary interaction and generic momentum fields is non-local so the effect of the interaction is no longer captured by a global SU(2) rotation. The following calculations therefore only apply to the subset of bulk amplitudes where all the operator insertions involve discrete fields.

A general bulk discrete primary field is given by (2.10). Insert one of these fields in the upper half-plane and use the method of images so that $\bar{\psi}_{\bar{\jmath}\bar{m}}(\bar{z}) \to \psi_{\bar{\jmath}\bar{m}}(z^*)$. As before, the boundary interaction acts on the image operator in the lower-half-plane by the global SU(2) rotation $U(g)$ given in (1.9). The original right-moving discrete field was a component of a rank $\bar{\jmath}$ irreducible tensor operator, so the rotated image is

$$\widetilde{\psi}_{\bar{\jmath}\bar{m}}(z^*) = \sum_{\bar{m}'=-\bar{\jmath}}^{\bar{\jmath}} \mathcal{D}^{\bar{\jmath}}_{\bar{m},\bar{m}'}(g)\psi_{\bar{\jmath}\bar{m}'}(z^*)\,, \tag{2.15}$$

with the rotation coefficient given by

$$\mathcal{D}^{\bar{\jmath}}_{\bar{m},\bar{m}'}(g) = \langle\bar{\jmath},\bar{m}|U(g)|\bar{\jmath},\bar{m}'\rangle = \langle\bar{\jmath},\bar{m}|e^{\pi i(gJ_+ + \bar{g}J_-)}|\bar{\jmath},\bar{m}'\rangle\,, \tag{2.16}$$

where $|j,m\rangle$ are standard SU(2) states. For simplicity we will choose g to be a real number in the following. In that case the coefficients $\mathcal{D}^{j}_{m,m'}(g)$ can be expressed in terms of $\cos\pi g$ and $\sin\pi g$. A general formula is given in appendix A, along with some explicit examples.

A bulk n-point function involving bulk discrete primary fields in the interacting theory can therefore be expressed in terms of $2n$-point functions of holomorphic discrete fields in the free theory along with SU(2) rotation coefficients,

$$\begin{aligned}\langle\Psi_{j_1m_1,\bar{j}_1\bar{m}_1}(z_1,\bar{z}_1)\ldots\Psi_{j_nm_n,\bar{j}_n\bar{m}_n}(z_n,\bar{z}_n)\rangle_g &= \\ = \sum_{\bar{m}'_i=-\bar{j}_i}^{\bar{j}_i}\mathcal{D}^{\bar{j}_1}_{\bar{m}_1,\bar{m}'_1}(g)\ldots\mathcal{D}^{\bar{j}_n}_{\bar{m}_n,\bar{m}'_n}(g)\,&\langle\psi_{j_1m_1}(z_1)\psi_{\bar{j}_1\bar{m}'_1}(z_1^*)\ldots\rangle\,.\end{aligned} \tag{2.17}$$

The only dependence on the boundary coupling is through the rotation coefficients that appear in (2.17).

Below we will illustrate this general result by considering one- and two-point functions of bulk fields, but let us first note that these bulk correlators have another equivalent representation. When implementing the method of rotated images we chose to deform the integration contours of the boundary currents into the lower half-plane but we could just as well have moved them into the upper half-plane instead. This results in the inverse SU(2) rotation acting on the left-moving part of each bulk field,

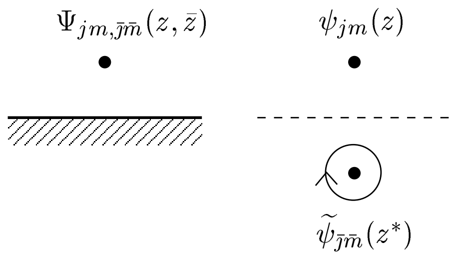

Figure 1: The method of rotated images applied to the one-point function of a bulk discrete primary field.

$$\widetilde{\psi}_{jm}(z) = \sum_{m'=-j}^{j}\mathcal{D}^{j}_{m,m'}(-g)\psi_{jm'}(z)\,, \tag{2.18}$$

and the following expression for the bulk n-point function,

$$\begin{aligned}&\langle\Psi_{j_1m_1,\bar{j}_1\bar{m}_1}(z_1,\bar{z}_1)\ \ldots\ \Psi_{j_nm_n,\bar{j}_n\bar{m}_n}(z_n,\bar{z}_n)\rangle_g = \\ &\qquad = \sum_{m'_i=-j_i}^{j_i}\mathcal{D}^{j_1}_{m_1,m'_1}(-g)\ldots\mathcal{D}^{j_n}_{m_n,m'_n}(-g)\,\langle\psi_{j_1m'_1}(z_1)\psi_{\bar{j}_1\bar{m}_1}(z_1^*)\ldots\rangle\,.\end{aligned} \tag{2.19}$$

At first glance this expression looks different from (2.17) but the two must be equivalent. We will verify this for bulk one-point functions below and we have also checked that it works for a number of examples involving higher-point functions.

2.3.1 Bulk one-point functions

In the free theory with Neumann boundary conditions momentum conservation implies that the only bulk operator that has a non-vanishing one-point function is the unit operator. The periodic boundary interaction can absorb momenta $p = n/\sqrt{2}$, with $n\in\mathbb{Z}$, and so any bulk operator that carries such a momentum will have a non-vanishing one-point function.

These are precisely the discrete bulk operators and the method of rotated images gives

$$\langle \Psi_{jm,\bar{j}\bar{m}}(z,\bar{z})\rangle_g = \sum_{\bar{m}'=-\bar{j}}^{\bar{j}} \mathcal{D}^{\bar{j}}_{\bar{m},\bar{m}'}(g)\langle \psi_{jm}(z)\psi_{\bar{j}\bar{m}'}(z^*)\rangle \,. \tag{2.20}$$

Conformal invariance requires the scaling dimension of the two chiral operators to be the same, i.e. $\bar{j}^2 = j^2$, while momentum conservation in the free chiral theory requires $\bar{m}' = -m$. The bulk one-point function is therefore given by

$$\begin{aligned}\langle \Psi_{jm,\bar{j}\bar{m}}(z,\bar{z})\rangle_g &= \delta_{j,\bar{j}}\,\mathcal{D}^{j}_{\bar{m},-m}(g)\,\langle \psi_{j,m}(z)\psi_{j,-m}(\bar{z})\rangle \\ &= (-1)^{N[j,m]}\,\delta_{j,\bar{j}}\,\frac{\mathcal{D}^{j}_{\bar{m},-m}(g)}{(z-z^*)^{2j^2}}\,,\end{aligned} \tag{2.21}$$

where $N[j,m]$ is an integer that depends on the normalization conventions for the discrete fields. We can relate the $N[j,m]$ for different values of m belonging to any given j as follows. The chiral two-point function in (2.21) involves two discrete fields whose operator product includes ψ_{00}, also known as the unit operator, with an OPE coefficient that is proportional to the Clebsch-Gordan coefficient $C^{00}_{j,m;j,-m}$. For a given value of j these particular Clebsch-Gordan coefficients all have the same magnitude but alternate in sign with m,

$$C^{00}_{j,m;j,-m} = (-1)^{j-m} C^{00}_{j,j;j,-j}\,. \tag{2.22}$$

These alternating signs are inherited by the $N[j,m]$ for a given j, leaving only $N[j,j]$ to be determined case by case.

If we instead move the integration contours of the boundary currents into the upper half-plane we obtain an answer that looks somewhat different,

$$\langle \Psi_{jm,\bar{j}\bar{m}}(z,\bar{z})\rangle_g = (-1)^{N[j,-\bar{m}]}\,\delta_{j,\bar{j}}\,\frac{\mathcal{D}^{j}_{m,-\bar{m}}(-g)}{(z-z^*)^{2j^2}}\,, \tag{2.23}$$

but due to the property, $\mathcal{D}^{j}_{m,-\bar{m}}(-g) = (-1)^{-(m+\bar{m})}\mathcal{D}^{j}_{\bar{m},-m}(g)$, which follows from the general formula for the rotation coefficients given in appendix A, and the fact that $N[j,-\bar{m}] = (-1)^{m+\bar{m}}N[j,m]$, which follows from (2.22), the two expressions for the bulk one-point function are in fact exactly the same.

2.3.2 Bulk two-point functions

A two-point function of bulk discrete fields involves rotation coefficients and four-point functions of holomorphic discrete fields, which in turn can be expressed in terms of OPE coefficients and holomorphic conformal blocks. Let's consider the two-point function of generic bulk discrete primaries,

$$\langle \Psi_{j_1m_1,\bar{j}_1\bar{m}_1}(z,\bar{z})\Psi_{j_2m_2,\bar{j}_2\bar{m}_2}(w,\bar{w})\rangle_g\,, \tag{2.24}$$

which according to the method of rotated images has the chiral form

$$\sum_{\bar{m}_1'=-\bar{j}_1}^{\bar{j}_1}\sum_{\bar{m}_2'=-\bar{j}_2}^{\bar{j}_2} \mathcal{D}^{\bar{j}_1}_{\bar{m}_1,\bar{m}_1'}\mathcal{D}^{\bar{j}_2}_{\bar{m}_2,\bar{m}_2'}\langle \psi_{j_1m_1}(z)\psi_{\bar{j}_1\bar{m}_1'}(z^*)\psi_{j_2m_2}(w)\psi_{\bar{j}_2\bar{m}_2'}(w^*)\rangle\,. \tag{2.25}$$

The chiral four-point functions can in principle be evaluated by using the free holomorphic propagator (2.2).

We can express any bulk two-point function of discrete primary fields in terms of chiral SU(2) conformal blocks. Each chiral four-point function that appears in (2.25) can be reduced to a sum over two-point functions by first applying the holomorphic OPE (2.7) to $\psi_{j_1 m_1}(z)$ and the image field $\psi_{\bar{\jmath}_1 \bar{m}'_1}(z^*)$ and separately to $\psi_{j_2 m_2}(w)$ and $\psi_{\bar{\jmath}_2 \bar{m}'_2}(w^*)$. The end result can be expressed in terms of conformal blocks $\mathcal{F}^{(j_1 m_1)(\bar{\jmath}_1 \bar{m}'_1)}_{(j_2 m_2)(\bar{\jmath}_2 \bar{m}'_2)}(J|\eta)$ with intermediate states belonging to the conformal family of highest weight J^2, where $|j_1 - \bar{\jmath}_1| \le J \le j_1 + \bar{\jmath}_1$. Each conformal block depends on the position of the operators only through the anharmonic ratio

$$\eta = \frac{(z-z^*)(w-w^*)}{(z-w^*)(w-z^*)}\,, \tag{2.26}$$

which is real valued, $0 \le \eta \le 1$, for all z, w in the upper half plane. The sum over conformal blocks is accompanied by a prefactor involving powers of $(z-w)$, $(z-w^*)$, etc., that has the required scaling dimension $j_1^2 + \bar{\jmath}_1^2 + j_2^2 + \bar{\jmath}_2^2$. We use standard conventions for conformal blocks [23]. The form of the prefactor is fixed by the mapping that takes the standard arguments of the fields in the four-point function $(0, \eta, 1, \infty)$ to the ones we are using $(\bar{z}, z, w, \bar{w})$. The bulk two-point function (2.24) is then given by

$$\begin{aligned}&(z-w^*)^{-2j_1^2}(w-z^*)^{-j_1^2-\bar{\jmath}_1^2-j_2^2+\bar{\jmath}_2^2}(w-w^*)^{j_1^2+\bar{\jmath}_1^2-j_2^2-\bar{\jmath}_2^2}(z^*-w^*)^{j_1^2-\bar{\jmath}_1^2+j_2^2-\bar{\jmath}_2^2}\times\\ &\times\sum_{\bar{m}'_1,\bar{m}'_2}\mathcal{D}^{\bar{\jmath}_1}_{\bar{m}_1,\bar{m}'_1}\mathcal{D}^{\bar{\jmath}_2}_{\bar{m}_2,\bar{m}'_2}\sum_J A^{J(m_1+\bar{m}'_1)}_{j_1 m_1,\bar{\jmath}_1\bar{m}'_1}A^{J(m_2+\bar{m}'_2)}_{j_2 m_2,\bar{\jmath}_2\bar{m}'_2}\mathcal{F}^{(j_1 m_1)(\bar{\jmath}_1 \bar{m}'_1)}_{(j_2 m_2)(\bar{\jmath}_2 \bar{m}'_2)}(J|\eta)\,.\end{aligned} \tag{2.27}$$

The conformal block $\mathcal{F}^{(j_1 m_1)(\bar{\jmath}_1 \bar{m}'_1)}_{(j_2 m_2)(\bar{\jmath}_2 \bar{m}'_2)}(J|\eta)$ is zero if $m_1 + \bar{m}'_1 + m_2 + \bar{m}'_2 \neq 0$ but since we sum over $\bar{m}'_1$ and $\bar{m}'_2$ there will in general be a non-vanishing contribution to the sum in (2.27).

As an example of a bulk two-point function we take $j_i = \bar{\jmath}_i = \frac{1}{2}$ and $m_1 = \bar{m}_1 = \frac{1}{2}, m_2 = \bar{m}_2 = -\frac{1}{2}$. The bulk fields decompose as

$$\Psi_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}(z,\bar{z}) \;=\; \psi_{\frac{1}{2}\frac{1}{2}}(z)\;\bar{\psi}_{\frac{1}{2}\frac{1}{2}}(\bar{z})\,, \tag{2.28}$$

$$\Psi_{\frac{1}{2}(-\frac{1}{2}),\frac{1}{2}(-\frac{1}{2})}(w,\bar{w}) \;=\; \psi_{\frac{1}{2}(-\frac{1}{2})}(w)\;\bar{\psi}_{\frac{1}{2}(-\frac{1}{2})}(\bar{w})\,, \tag{2.29}$$

and the rotated image fields are

$$\tilde{\psi}_{\frac{1}{2}\frac{1}{2}}(z^*) \;=\; \sum_{m=\pm\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}_{\frac{1}{2},m}(g)\;\psi_{\frac{1}{2}m}(z^*) = \cos\pi g\;\psi_{\frac{1}{2}\frac{1}{2}}(z^*) + i\sin\pi g\;\psi_{\frac{1}{2}(-\frac{1}{2})}(z^*)\,,$$

$$\tilde{\psi}_{\frac{1}{2}(-\frac{1}{2})}(w^*) \;=\; \sum_{m=\pm\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}_{-\frac{1}{2},m}(g)\;\psi_{\frac{1}{2}m}(w^*) = i\sin\pi g\;\psi_{\frac{1}{2}\frac{1}{2}}(w^*) + \cos\pi g\;\psi_{\frac{1}{2}(-\frac{1}{2})}(w^*)\,.$$

There are four chiral four-point functions to be evaluated but only two of them conserve momentum and we get

$$\langle\Psi_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}(z,\bar{z})\Psi_{\frac{1}{2}(-\frac{1}{2}),\frac{1}{2}(-\frac{1}{2})}(w,\bar{w})\rangle_g =$$

$$
\begin{aligned}
&= \cos^2 \pi g \ \left\langle \psi_{\frac{1}{2}\frac{1}{2}}(z)\ \psi_{\frac{1}{2}\frac{1}{2}}(z^*)\ \psi_{\frac{1}{2}(-\frac{1}{2})}(w)\ \psi_{\frac{1}{2}(-\frac{1}{2})}(w^*) \right\rangle - \\
&\quad - \sin^2 \pi g \ \left\langle \psi_{\frac{1}{2}\frac{1}{2}}(z)\ \psi_{\frac{1}{2}(-\frac{1}{2})}(z^*)\ \psi_{\frac{1}{2}(-\frac{1}{2})}(w)\ \psi_{\frac{1}{2}\frac{1}{2}}(w^*) \right\rangle \\
&= \frac{\cos^2 \pi g}{|z-w|} \left[\frac{(z-z^*)(w-w^*)}{(z-w^*)(z^*-w)} \right]^{1/2} - \frac{\sin^2 \pi g}{|z-w|} \left[\frac{(z-z^*)(w-w^*)}{(z-w^*)(z^*-w)} \right]^{-1/2} . \qquad (2.30)
\end{aligned}
$$

This can be reexpressed to match the general formula (2.27) as follows,

$$
\begin{aligned}
&\langle \Psi_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}(z,\bar{z}) \Psi_{\frac{1}{2}(-\frac{1}{2}),\frac{1}{2}(-\frac{1}{2})}(w,\bar{w}) \rangle_g = \\
&= \frac{1}{(z-w^*)^{1/2}(w-z^*)^{1/2}} \left(\cos^2 \pi g \ \left[\frac{\eta}{1-\eta} \right]^{1/2} - \sin^2 \pi g \ \left[\frac{1}{\eta(1-\eta)} \right]^{1/2} \right) \qquad (2.31)
\end{aligned}
$$

revealing the conformal block structure in this simple example.

2.3.3 Crossing symmetry

The bulk two-point functions in the previous subsection can be computed in another way, namely by first taking the OPE of the two holomorphic fields and separately the OPE of the two anti-holomorphic fields, and then applying the method of rotated images to the result of the anti-holomorphic OPE. The chiral four-point function again reduces to a sum of chiral two-point functions, as indicated in figure 2, but since the rotated image operators are now in different representations of the SU(2) algebra, the conformal blocks that appear are not the same as before. Crossing symmetry is the statement that the two methods give the same answer and in this subsection we verify that the method of rotated images preserves this symmetry. This provides a non-trivial check on our prescription.

Crossing symmetry for $\langle \psi_{j_1 m_1}(z) \psi_{\bar{j}_1 \bar{m}_1}(z^*) \psi_{j_2 m_2}(w) \psi_{\bar{j}_2 \bar{m}_2}(w^*) \rangle$ in the free theory can be expressed as

$$
\begin{aligned}
&\sum_{J_p} A^{J_p(m_1+\bar{m}_1)}_{j_1 m_1,\bar{j}_1\bar{m}_1} A^{J_p(m_2+\bar{m}_2)}_{j_2 m_2,\bar{j}_2\bar{m}_2} \mathcal{F}^{(j_1 m_1)(\bar{j}_1\bar{m}_1)}_{(j_2 m_2)(\bar{j}_2\bar{m}_2)}(J_p|\eta) = \qquad (2.32) \\
&\qquad = \sum_{J_q} A^{J_q(m_1+m_2)}_{j_1 m_1,j_2 m_2} A^{J_q(\bar{m}_1+\bar{m}_2)}_{\bar{j}_1\bar{m}_1,\bar{j}_2\bar{m}_2} \mathcal{F}^{(j_1 m_1)(j_2 m_2)}_{(\bar{j}_1\bar{m}_1)(\bar{j}_2\bar{m}_2)}(J_q|1-\eta).
\end{aligned}
$$

The prefactor that accompanies the conformal blocks in (2.27) is the same in both channels and therefore cancels in the crossing relation. With the boundary interaction turned on, we rotate $\psi_{\bar{j}_1\bar{m}_1}$ and $\psi_{\bar{j}_2\bar{m}_2}$ on the left hand side, but the product field $\psi_{J_q M_q}$, with $M_q =$

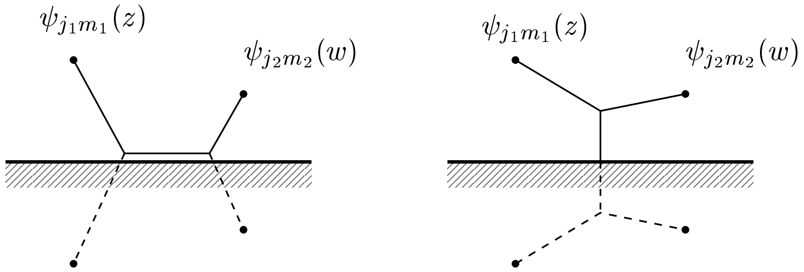

Figure 2: Crossing symmetry.

$\bar{m}_1+\bar{m}_2$, on the right hand side. The requirement of crossing symmetry then becomes

$$\sum_{\bar{m}'_1,\bar{m}'_2}\mathcal{D}^{\bar{j}_1}_{\bar{m}_1,\bar{m}'_1}\mathcal{D}^{\bar{j}_2}_{\bar{m}_2,\bar{m}'_2}\sum_{J_p}A^{J_p(m_1+\bar{m}'_1)}_{j_1m_1,\bar{j}_1\bar{m}'_1}A^{J_p(m_2+\bar{m}'_2)}_{j_2m_2,\bar{j}_2\bar{m}'_2}\mathcal{F}^{(j_1m_1)(\bar{j}_1\bar{m}'_1)}_{(j_2m_2)(\bar{j}_2\bar{m}'_2)}(J_p|\eta)= \tag{2.33}$$
$$=\sum_{J_q}A^{J_q(m_1+m_2)}_{j_1m_1,j_2m_2}A^{J_q(\bar{m}_1+\bar{m}_2)}_{\bar{j}_1\bar{m}_1,\bar{j}_2\bar{m}_2}\mathcal{D}^{J_q}_{(\bar{m}_1+\bar{m}_2),(-m_1-m_2)}\mathcal{F}^{(j_1m_1)(j_2m_2)}_{(\bar{j}_1,-m_1)(\bar{j}_2,-m_2)}(J_q|1-\eta).$$

The conformal block $\mathcal{F}^{(j_1m_1)(j_2m_2)}_{(\bar{j}_1\bar{m}_1)(\bar{j}_2\bar{m}_2)}(J_q|1-\eta)$ does not depend on the m's individually, but only on the sums m_1+m_2 and $\bar{m}_1+\bar{m}_2$, and due to momentum conservation the conformal block vanishes unless $m_1+m_2=-(\bar{m}_1+\bar{m}_2)$. In the equation above we used $\bar{m}'_1=-m_1$ and $\bar{m}'_2=-m_2$ to express the last conformal block, but we could just as well have used other $\bar{m}$'s with the same sum.

To see that crossing symmetry is satisfied we start with the left hand side of equation (2.33), use the free crossing symmetry (2.32) on the inner sum and rewrite the product of the rotation coefficients using the relation [24, 25]

$$\mathcal{D}^{j_1}_{m_1,n_1}(g)\mathcal{D}^{j_2}_{m_2,n_2}(g)=\sum_{J=|j_1-j_2|}^{j_1+j_2}C^{J(m_1+m_2)}_{j_1m_1,j_2m_2}C^{J(n_1+n_2)}_{j_1n_1,j_2n_2}\mathcal{D}^{J}_{(m_1+m_2),(n_1+n_2)}(g)\,, \tag{2.34}$$

where the $C^{JM}_{j_1m_1,j_2m_2}$ are Clebsch-Gordan coefficients. The left hand side of (2.33) becomes

$$\sum_{\bar{m}'_1,\bar{m}'_2}\sum_{J}C^{J(\bar{m}_1+\bar{m}_2)}_{\bar{j}_1\bar{m}_1,\bar{j}_2\bar{m}_2}C^{J(\bar{m}'_1+\bar{m}'_2)}_{\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2}\mathcal{D}^{J}_{(\bar{m}_1+\bar{m}_2),(\bar{m}'_1+\bar{m}'_2)}\times$$
$$\times\sum_{J_q}A^{J_q(m_1+m_2)}_{j_1m_1,j_2m_2}A^{J_q(\bar{m}'_1+\bar{m}'_2)}_{\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2}\mathcal{F}^{(j_1m_1)(j_2m_2)}_{(\bar{j}_1\bar{m}'_1)(\bar{j}_2\bar{m}'_2)}(J_q|1-\eta). \tag{2.35}$$

Now we recall from equation (2.8) that $A^{JM}_{jm,j'm'}=f(j,j',J)C^{JM}_{jm,j'm'}$ and therefore

$$A^{J_q(\bar{m}'_1+\bar{m}'_2)}_{\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2}=A^{J(\bar{m}_1+\bar{m}_2)}_{\bar{j}_1\bar{m}_1,\bar{j}_2\bar{m}_2}\frac{C^{J_q(\bar{m}'_1+\bar{m}'_2)}_{\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2}}{C^{J(\bar{m}_1+\bar{m}_2)}_{\bar{j}_1\bar{m}_1,\bar{j}_2\bar{m}_2}}\frac{f(\bar{j}_1,\bar{j}_2,J_q)}{f(\bar{j}_1,\bar{j}_2,J)}. \tag{2.36}$$

Making this substitution in (2.35), the $C^{J(\bar{m}_1+\bar{m}_2)}_{\bar{j}_1\bar{m}_1,\bar{j}_2\bar{m}_2}$ cancels and we get

$$\sum_{J,J_q}{\sum_{\bar{m}'_1,\bar{m}'_2}}'C^{J_q(\bar{m}'_1+\bar{m}'_2)}_{\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2}C^{J(\bar{m}'_1+\bar{m}'_2)}_{\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2}\mathcal{D}^{J}_{(\bar{m}_1+\bar{m}_2),(-m_1-m_2)}\frac{f(J_q)}{f(J)}\times$$
$$\times A^{J_q(m_1+m_2)}_{j_1m_1,j_2m_2}A^{J(\bar{m}_1+\bar{m}_2)}_{\bar{j}_1\bar{m}_1,\bar{j}_2\bar{m}_2}\mathcal{F}^{(j_1m_1)(j_2m_2)}_{(\bar{j}_1,-m_1)(\bar{j}_2,-m_2)}(J_q|1-\eta) \tag{2.37}$$

where we have replaced $\bar{m}'_1+\bar{m}'_2\to-m_1-m_2$ in the rotation coefficient and $\bar{m}'_1\to-m_1$ and $\bar{m}'_2\to-m_2$ in the conformal block. The prime on the sum over $\bar{m}'_1$ and $\bar{m}'_2$ means that we restrict the sum to $\bar{m}'_1+\bar{m}'_2=-m_1-m_2$. This can be done because the conformal block vanishes for other values of $\bar{m}'_1+\bar{m}'_2$. The sum over the Clebsch-Gordan coefficients is

$${\sum_{\bar{m}'_1,\bar{m}'_2}}'C^{J_q(\bar{m}'_1+\bar{m}'_2)}_{\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2}C^{J(\bar{m}'_1+\bar{m}'_2)}_{\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2}=\sum_{\bar{m}'_1,\bar{m}'_2}\langle J_qM|\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2\rangle\langle\bar{j}_1\bar{m}'_1,\bar{j}_2\bar{m}'_2|JM\rangle$$

$$= \langle J_q M | J M \rangle = \delta(J_q - J)\,, \tag{2.38}$$

where $M = -m_1 - m_2$ and we have used the completeness relation for $|\bar\jmath_1 \bar m_1', \bar\jmath_2 \bar m_2'\rangle$. Finally, we carry out the sum over J to arrive at

$$\sum_{J_q} A^{J_q(m_1+m_2)}_{j_1 m_1, j_2 m_2} A^{J_q(\bar m_1+\bar m_2)}_{\bar\jmath_1 \bar m_1, \bar\jmath_2 \bar m_2} \mathcal{D}^{J_q}_{(\bar m_1+\bar m_2),(-m_1-m_2)} \mathcal{F}^{(j_1 m_1)(j_2 m_2)}_{(\bar\jmath_1,-m_1)(\bar\jmath_2,-m_2)}(J_q|1-\eta) \tag{2.39}$$

which is exactly the right hand side of equation (2.33).

3. Boundary fields

When a bulk field approaches the boundary at $z = \bar z$, new divergences appear that are not removed by the bulk normal ordering. This is a general feature of conformal field theories with boundaries and signals the presence of so-called *boundary fields* in the theory [26]. A bulk field approaching the boundary has an operator product expansion,

$$\Psi_{h,\bar h}(z,\bar z) = \sum_i \frac{B^i_{h,\bar h}}{(z-\bar z)^{h+\bar h-\Delta_i}}\, \Psi^B_i(x)\,, \tag{3.1}$$

where $x = (z+\bar z)/2$. The $\Psi^B_i(x)$ are boundary fields with boundary scaling dimensions Δ_i and the $B^i_{h,\bar h}$ are called bulk-to-boundary operator product coefficients.

In addition to the bulk-to-boundary OPE, the boundary fields form an operator product algebra amongst themselves,

$$\Psi^B_i(x)\, \Psi^B_j(x') = \sum_k \frac{C_{ijk}}{(x-x')^{\Delta_i+\Delta_j-\Delta_k}}\, \Psi^B_k(x')\,. \tag{3.2}$$

The boundary OPE coefficients C_{ijk} and the bulk-to-boundary OPE coefficients $B^i_{h,\bar h}$, along with the boundary scaling dimensions Δ_i, are characteristic data of a given boundary conformal field theory.

In radial quantization boundary operators correspond to open string states, defined on a semi-circle enclosing the insertion point on the boundary. The neighborhood of the insertion point can be mapped by a conformal transformation to the strip, as shown in figure 3.

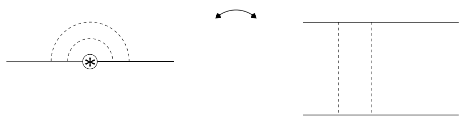

Figure 3: Correspondence between boundary operators and open string states via map between the halfplane and infinite strip.

The boundary scaling dimension Δ_i of a boundary operator $\Psi^B_i(x)$ is given by the energy eigenvalue of the corresponding open string state, where the open-string hamiltonian is the L_0 generator of the Virasoro algebra that is preserved by the conformally invariant boundary conditions. With this in mind, we will examine the open string spectrum of the interacting $c = 1$ model defined by (1.1) in detail in section 4, but first let us look at how all of this works in the free theory.

3.1 Free theory

In this case we can use the doubling trick to replace the right-moving part of the bulk operator in (3.1) by its left-moving image,

$$\Psi_{h,\bar{h}}(z,\bar{z}) \to \psi_h(z)\psi_{\bar{h}}(z^*)\,, \tag{3.3}$$

and view the approach to the boundary as two left-moving fields moving towards the real axis from opposite directions. We can then work out the operator product using free-field Wick contractions and the holomorphic propagator (2.2). Consider a non-compact boson and let a bulk primary field at generic momentum approach the boundary. The chiral OPE between its left-moving part and the left-moving image of the right-moving part is

$$\begin{aligned} e^{ik\phi(z)}e^{ik\phi(z^*)} &= \frac{e^{2ik\phi(\frac{z+z^*}{2})}}{(z-z^*)^{-k^2}} + \cdots \\ &= \frac{e^{ik\Phi(x)}}{(z-z^*)^{-k^2}} + \cdots \end{aligned} \tag{3.4}$$

where $x = (z+z^*)/2$ and the boundary scalar field $\Phi(x)$ is related to the left-moving field as before by $\Phi(x) = 2\phi(z)|_{z=z^*}$. The free-field propagator on the boundary is

$$\langle \Phi(x)\Phi(x')\rangle = -4\log|x-x'|\,, \tag{3.5}$$

and thus the boundary scaling dimension of $\Psi^B_k(x) = e^{ik\Phi(x)}$ is $\Delta_k = 2k^2$. Since the bulk scaling dimension of $e^{ik\phi(z)}$ is $h = k^2/2$, we see that (3.4) is consistent with the general form (3.1) and we can read off $B^k_{h,\bar{h}} = 1$ for the bulk-to-boundary OPE coefficient. The boundary OPE for primary fields of the form $e^{ik\Phi(x)}$ is given by

$$e^{ik_1\Phi(x)}e^{ik_2\Phi(x')} = \frac{e^{i(k_1+k_2)\Phi(x')}}{|x-x'|^{4k_1k_2}} + \cdots\,. \tag{3.6}$$

This has the form of (3.2) with $C_{k_1,k_2,k_3} = \delta_{k_1+k_2+k_3,0}$.

When a bulk discrete field approaches the boundary we get

$$\psi_{jm}(z)\psi_{\bar{j}\bar{m}}(z^*) = \sum_{J=|j-\bar{j}|}^{j+\bar{j}}\ \sum_{M=-J}^{J} \frac{B^{JM}_{jm,\bar{j}\bar{m}}}{(z-z^*)^{j^2+\bar{j}^2-J^2}}\Psi_{JM}(x) + \cdots\,, \tag{3.7}$$

where the $\Psi_{JM}(x)$ are primary fields on the boundary, which we will refer to as *discrete boundary fields*. The discrete boundary fields come in SU(2) multiplets just as the chiral discrete fields, but since $j-\bar{j} \in \mathbb{Z}$ only integer values of J are allowed on the boundary. The currents of the boundary SU(2) algebra are given by

$$J_\pm = e^{\pm i\Phi/\sqrt{2}} \qquad \text{and} \qquad J_3 = \frac{i}{2\sqrt{2}}\frac{d\Phi}{dx}\,. \tag{3.8}$$

The discrete boundary fields can be easily obtained as follows. Work in the theory at the self-dual radius and let the purely holomorphic bulk discrete primary field $\Psi_{jm,00}(z,\bar{z}) = \psi_{jm}(z)\bar{\psi}_{00}(\bar{z})$ approach the boundary. Since $\bar{\psi}_{00}(\bar{z})$ is the unit operator the bulk-to-boundary OPE only contains one primary on the right hand side and we find that

$$\Psi_{JM}(x) = \psi_{JM}(z)|_{z=x} \,. \tag{3.9}$$

The discrete boundary field Ψ_{JM} is thus made from $e^{iM\,\Phi/\sqrt{2}}$ times a polynomial in derivatives of Φ that is obtained from the one that appears in $\psi_{JM}(z)$ by the replacement $\partial\Phi \to \Phi'/2$. The boundary scaling dimension of Ψ_{JM} is J^2. This construction of discrete boundary fields is in the theory at the self-dual radius but the resulting Ψ_{JM} boundary fields also exist at other radii. They form a closed algebra under the boundary OPE (3.2), with contractions carried out using the free boundary propagator (3.5).

It follows from the above construction that in the free theory the bulk-to-boundary operator product coefficients $B^{JM}_{jm,\bar{j}\bar{m}}$ for discrete primary fields are given precisely by the chiral OPE coefficients $A^{JM}_{jm,\bar{j}\bar{m}}$ in (2.7) for integer J. Finally, we note that the field Ψ_{JM} could equally well have been obtained from the anti-holomorphic bulk field $\Psi_{00,JM}$.

3.2 Interacting theory

The boundary fields are modified when the interaction (1.7) is turned on. The boundary OPE can no longer be evaluated using the free field boundary propagator (3.5) and the boundary primary field $\Psi_p(x)$ carrying generic momentum p no longer has the simple form $e^{ik\Phi(x)}$. This is evident from the change in the scaling dimensions of boundary primary operators at generic momentum. The boundary scaling dimensions correspond to L_0 eigenvalues of the corresponding open string states and the open string spectrum undergoes a non-trivial "flow" as a function of the boundary coupling g. Bands are formed with gaps between them that grow wider with increasing coupling strength [12]. We discuss the open string spectrum in detail in section 4. The remaining data that defines the interacting boundary conformal field theory, i.e. the bulk-to-boundary OPE and the boundary OPE, will also depend on coupling g. Conformal invariance places restrictions on low-order correlation functions of boundary operators, but beyond that little is known.

At the momenta carried by the boundary discrete fields the situation is simplified. As we will see in section 4, the dimension of discrete boundary fields Ψ_{JM} does not flow when the boundary interaction is turned on but remains at J^2. This enables us to write a discrete boundary field in the interacting theory Ψ^g_{JM} as a linear combination of the free boundary fields in the SU(2) representation with the same J value,

$$\Psi^g_{JM}(x) = \sum_{M'=-J}^{J} h^J_{MM'}(g)\Psi^0_{JM'}(x) \,. \tag{3.10}$$

The bulk-to-boundary OPE for discrete fields at non-zero boundary coupling is given by

$$\Psi_{jm,\bar{j}\bar{m}}(z,\bar{z}) = \sum_{J=|j-\bar{j}|}^{j+\bar{j}} \sum_{M=-J}^{J} \frac{B^{JM}_{jm,\bar{j}\bar{m}}(g)}{(z-z^*)^{j^2+\bar{j}^2-J^2}} \Psi^g_{JM}(x) + \cdots \,. \tag{3.11}$$

The bulk discrete field is unaffected by the boundary physics but in general both the bulk-to-boundary OPE coefficients and the boundary discrete fields may be expected to depend on g.

Now we apply the method of rotated images to the bulk primary field on the left hand side of (3.11) and then use the free theory bulk-to-boundary OPE (3.7) to obtain,

$$\psi_{jm}(z)\tilde{\psi}_{\bar{\jmath}\bar{m}}(z^*) = \sum_{\bar{m}'=-\bar{\jmath}}^{\bar{\jmath}} \sum_{J=|j-\bar{\jmath}|}^{j+\bar{\jmath}} \sum_{M=-J}^{J} \mathcal{D}^{\bar{\jmath}}_{\bar{m},\bar{m}'}(g) \frac{B^{JM}_{jm,\bar{\jmath}\bar{m}'}(0)}{(z-z^*)^{j^2+\bar{\jmath}^2-J^2}} \Psi^0_{JM}(x) + \cdots . \tag{3.12}$$

The right hand side is to equal that of (3.11) and thus, by inserting the expansion (3.10) for $\Psi^g_{JM}(x)$, we obtain algebraic equations that relate the bulk-to-boundary OPE coefficients $B^{JM}_{jm,\bar{\jmath}\bar{m}}(g)$ and the expansion coefficients $h^J_{MM'}(g)$,

$$\sum_{M'=-J}^{J} B^{JM'}_{jm,\bar{\jmath}\bar{m}}(g)\, h^J_{M'M}(g) = \sum_{\bar{m}'=-\bar{\jmath}}^{\bar{\jmath}} \mathcal{D}^{\bar{\jmath}}_{\bar{m},\bar{m}'}(g)\, B^{JM}_{jm,\bar{\jmath}\bar{m}'}(0) . \tag{3.13}$$

Unfortunately, there are not enough equations to determine all the coefficients, so we need further input.

It seems reasonable to define the discrete boundary field $\Psi^g_{JM}(x)$ in the interacting theory in the same fashion as in the free theory, i.e. as the boundary primary field that arises when we let the purely holomorphic bulk field $\Psi_{JM,00}$ approach the boundary. If we then apply the method of rotated images, we find that the global SU(2) rotation acts trivially on the unit operator $\psi_{00}(z^*)$ and the discrete boundary field is actually the same as in the free theory,

$$h^J_{MM'}(g) = \delta_{M,M'} . \tag{3.14}$$

It follows that all boundary correlators of discrete fields will be independent of the boundary coupling g. This is a surprisingly strong result and certainly only applies to the discrete boundary fields. At generic momentum the boundary correlators depend on g in a non-trivial way.

When applying the method of rotated images to $\Psi_{JM,00}$ we deformed the integration contours of the boundary currents into the lower half-plane where they encountered only $\psi_{00}(z^*)$. We could equally well have moved the contours into the upper half-plane, leading to

$$h^J_{MM'} = \mathcal{D}^J_{MM'}(-g) . \tag{3.15}$$

In this case the algebraic relation (3.13) between $h^J_{MM'}(g)$ and the bulk-to-boundary OPE coefficients is modified,

$$\sum_{M'=-J}^{J} B^{JM'}_{jm,\bar{\jmath}\bar{m}}(g)\, h^J_{M'M}(g) = \sum_{m'=-j}^{j} \mathcal{D}^j_{m,m'}(-g)\, B^{JM}_{jm',\bar{\jmath}\bar{m}}(0) , \tag{3.16}$$

and the OPE coefficients that solve them are not the same as for (3.13). The new prescription nevertheless leads to the same boundary correlators as before. This is because (3.15) amounts to the same g-dependent SU(2) rotation acting on all the discrete boundary fields of the free theory, which leaves their correlation functions unchanged,

$$\langle \Psi^g_{J_1M_1}(x_1) \ldots \Psi^g_{J_nM_n}(x_n)\rangle =$$

$$= \sum_{M_i'=-J_i}^{J_i} \mathcal{D}^{J_1}_{M_1 M_1'}(g) \ldots \mathcal{D}^{J_n}_{M_n M_n'}(g) \langle \Psi^0_{J_1 M_1'}(x_1) \ldots \Psi^0_{J_n M_n'}(x_n) \rangle$$
$$= \langle \Psi^0_{J_1 M_1}(x_1) \ldots \Psi^0_{J_n M_n}(x_n) \rangle \,. \tag{3.17}$$

We could also have defined $\Psi_{JM}(x)$ as the boundary field obtained when we let the anti-holomorphic bulk field $\Psi_{00,JM}$ approach the boundary. In the free theory this definition is equivalent to the one based on a purely holomorphic bulk field and for correlation functions involving discrete boundary fields this remains true in the interacting theory, they are independent of g with either definition. As we will see in section 5.2 below, however, mixed amplitudes containing both bulk and boundary fields will in general depend on which definition we use.

3.3 Boundary condition changing fields

We can also consider more general boundary fields which change the boundary conditions where they are inserted into correlation functions. The map from the upper half-plane to the strip reveals that the corresponding open string states are subject to different boundary conditions at the two endpoints, as indicated in figure 4.

In the theory at hand, boundary conditions are labelled by the boundary coupling g which multiplies the periodic boundary potential in (1.1). A boundary condition changing operator changes g to g' at the insertion point on the boundary and thus corresponds to open string endpoints interacting with boundary potentials of different strength g and g'. We obtain the spectrum of such strings in section 4.2 below.

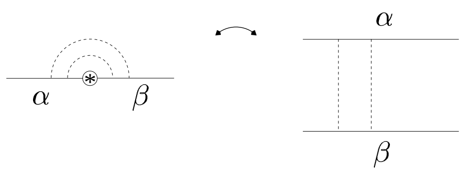

Figure 4: Boundary condition changing fields correspond to open strings with different boundary conditions at the two endpoints.

4. The open string spectrum

The scaling dimensions of boundary operators in a given boundary conformal field theory are given by the L_0 eigenvalues of the corresponding open string states. In this section we will mostly work with a non-compact boson, for which the open string spectrum is continuous. The discrete spectra obtained at finite boson radii are all included in the continuous spectrum and are obtained by retaining only those eigenvalues that correspond to allowed momenta in each case.

The information we are after is for example contained in the one-loop partition function in the open string channel,

$$\mathcal{Z} = \mathrm{Tr}\left[\exp(-\beta H_{\mathrm{open}})\right] . \tag{4.1}$$

For open strings with Neumann boundary conditions at both ends a standard calculation gives

$$\mathcal{Z}_{\mathrm{free}} = \frac{1}{\eta(\omega)} \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \omega^{p^2/2} \,, \tag{4.2}$$

where $\eta(\omega) = \omega^{1/24} \prod_{n=1}^{\infty}(1-\omega^n)$ is the Dedekind eta-function with $\omega = e^{-\pi\beta/\ell}$. Here ℓ is the parameter length of the open string. The p integral can be interpreted as a sum over Virasoro representations obtained from primary fields of the form $e^{ip\Phi(x)/\sqrt{2}}$.

4.1 Spectral flow and string band structure

The open string spectrum is modified in the presence of the periodic boundary potential (1.7). The interacting theory exhibits a non-trivial spectral flow that depends on the strength of the boundary coupling g. The system can be re-expressed in terms of free fermions as shown in [12]. The SU(2) currents are bi-linear in the fermions and the boundary interaction may be viewed as a localized mass term. The open string spectrum is then found by solving a straightforward eigenvalue problem for the fermions. We will not repeat the construction here but simply quote the result. In [12] both string endpoints were taken to interact with the same boundary potential, $g = g'$. In that case the partition function may be written

$$\mathcal{Z} = \frac{\sqrt{2}}{\eta(\omega)} \int_{-1/2}^{1/2} \frac{dk}{2\pi} \sum_{m=-\infty}^{\infty} \omega^{(\lambda+m)^2}\,, \tag{4.3}$$

where λ is related to the target space momentum $p = \sqrt{2}k$ by

$$\sin\pi\lambda = \cos\pi|g|\sin\pi k\,. \tag{4.4}$$

The value of λ lies within the range $-\frac{1}{2}+|g| \leq \lambda \leq \frac{1}{2}-|g|$ and the energy eigenvalues of the highest-weight open string states are given by $\Delta = (\lambda(k)+m)^2$, with $m \in \mathbb{Z}$.

The spectrum is shown in the left-most graph in figure 5. It has split into bands with forbidden gaps in energy in between them. Note that the energy eigenvalues are invariant under $k \to k+n$ for any integer n. This is due to the fact that the boundary potential breaks translation invariance in the target space to a discrete subgroup. In the interacting system target space momentum p is therefore only conserved mod $\sqrt{2}$ in our units, and a shift of k by an integer can always be used to bring the momentum of a given operator into the so-called first Brillouin zone, $-\frac{1}{\sqrt{2}} \leq p \leq \frac{1}{\sqrt{2}}$. The spectra in figure 5 are displayed in an extended zone scheme, with the periodicity in momentum appearing explicitly. The boundary fields $V_{n,p}(x)$ that correspond to these open string states carry two labels. The integer $n \geq 0$ denotes the band, to which the state belongs, and p is the momentum within the first Brillouin zone.

The allowed values of $\lambda+m$ in (4.3) consist of bands of width $1-2|g|$ centered at every integer, with gaps of width $2|g|$ between. When $g=0$ we get $\lambda+m=k$ and the gaps disappear. This corresponds to free propagation of open strings with Neumann boundary conditions at both ends. Equation (4.4) gives $\sin\pi\lambda = 0$ when $|g| = 1/2$ and λ must then be an integer. This is the tight binding limit, the bands have zero width and we get Dirichlet boundary conditions confining each string end to sit at one of the minima of the boundary potential.[4] This also has an interpretation as a sequence of D-particles, evenly spaced along the Φ direction in target space.

[4]By looking at the boundary potential in (1.1) one might have expected tight binding to occur for $|g| \to \infty$

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In conventional condensed matter systems, band gaps get vanishingly small as the energy becomes large compared to the characteristic scale set by the potential. As a consequence of conformal invariance, the stringy bands discussed here behave rather differently. Both the individual bands and the gaps between them grow wider with increasing energy.

It follows from (4.4) that the scaling dimension of a discrete boundary field $\Psi_{JM}(x)$ does not undergo spectral flow. These fields carry momenta that correspond to $k \in \mathbb{Z}$, for which the right hand side of (4.4) is zero for any value of g. This observation plays a key role when we discuss the discrete boundary operators in the interacting theory.

When the boson is compactified we have to restrict to allowed momentum values and the energy bands are discretized accordingly. For a boson at the self-dual radius, we only have the discrete fields. Since their scaling dimensions are unaffected, the full open string spectrum is in fact independent of the boundary interaction in this case, as was first observed in [11].

It is interesting to note that energy bands also appear in the open string channel of the boundary conformal field theory that describes a free boson compactified on a circle, when the radius of the circle is irrational [27, 28], i.e. not a rational number times the self-dual radius $R_{\rm sd} = \sqrt{2}$. Like the present system, those theories admit a one-parameter family of boundary states that interpolate between Neumann and Dirichlet boundary conditions, but it is unclear to us how deep the parallels between the systems run.

4.2 Open strings coupled to two different boundaries

The band spectrum found in [12] is a special case of a more general structure. When we allow for boundary condition changing operators we have to consider also open strings where the boundary coupling takes different values g_1 and g_2 at the endpoints.

The fermion eigenvalue problem solved in [12] can easily be extended to cover this case also.[5] The only modification is to change g to g_1 in one of the boundary mass terms for the worldsheet fermions in equation (29) of that paper and g to g_2 in the other one. The relevant eigenvalue equation becomes

$$e^{-iN_1} e^{iN_2} e^{iN_3} \Psi = \Lambda \Psi$$

where

$$N_1 = \pi \begin{pmatrix} 0 & w g_1 \\ \bar{w} \bar{g}_1 & 0 \end{pmatrix}, \qquad N_2 = \pi \begin{pmatrix} 0 & g_2 \\ \bar{g}_2 & 0 \end{pmatrix}, \qquad N_3 = 2\pi k \sigma^3 \tag{4.5}$$

and $w = e^{-2\pi i k}$. For simplicity we assume that $g_1, g_2 \in \mathbb{R}$, with $g_1 \geq g_2$, in the following. We notice that

$$\det e^{-iN_1} e^{iN_2} e^{iN_3} = \exp\left(\operatorname{tr} e^{-iN_1} \operatorname{tr} e^{iN_2} \operatorname{tr} e^{iN_3}\right) = 1$$

rather than $|g| \to 1/2$. As discussed in [11, 20], this is a renormalization effect. There are divergences in the model and the formula (4.4) for the spectrum implicitly refers to a particular regularization and subtraction procedure which redefines the coupling constant according to (1.10).

[5] The spectrum can also be obtained by evaluating the appropriate one-loop partition function in the closed-string channel. This calculation is presented in appendix B.

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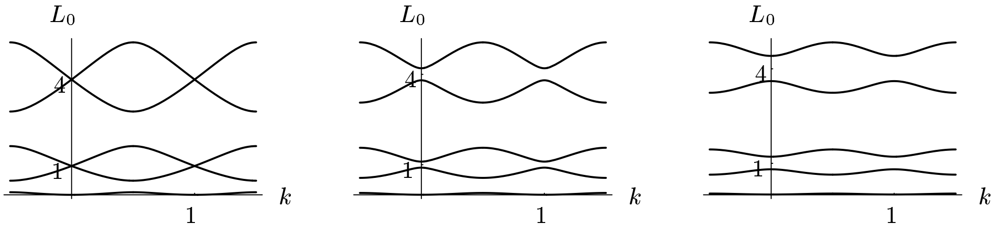

Figure 5: The first few bands of the energy spectrum for different values of the coupling constants. One is held fixed at $g_2 = 0.2$ but the other takes the values a) $g_1 = 0.2$, b) $g_1 = 0.3$, c) $g_1 = 0.4$.

and find that

$$\mathrm{tr}\left(e^{-iN_1} e^{iN_2} e^{iN_3}\right) = 2\cos(\pi g_1)\cos(\pi g_2)\cos(2\pi k) + 2\sin(\pi g_1)\sin(\pi g_2)\,. \tag{4.6}$$

The trace is real and lies between -2 and 2. The characteristic equation has a negative discriminant and real coefficients so the eigenvalues are complex conjugates of one another, $\Lambda_1 = \bar{\Lambda}_2$. Furthermore, $\Lambda_1\Lambda_2 = 1$ so we can write $\Lambda_{1,2} = e^{\pm 2\pi i\lambda}$. The trace is equal to the sum of the eigenvalues and the equation for λ becomes

$$2\cos(2\pi\lambda) = 2\cos(\pi g_1)\cos(\pi g_2)\cos(2\pi k) + 2\sin(\pi g_1)\sin(\pi g_2)$$

which can be rewritten as

$$\sin^2(\pi\lambda) = \sin^2(\pi g_-)\cos^2(\pi k) + \cos^2(\pi g_+)\sin^2(\pi k) \tag{4.7}$$

with $g_\pm = \frac{1}{2}(g_1 \pm g_2)$. Clearly this reduces to the previous result (4.4) as $g_1 \to g_2$ but when $g_1 \neq g_2$ there are important new features. In particular, there are additional gaps in the spectrum as shown in figure 5.

By varying either g_1 or g_2 from zero to one-half we effectively interpolate between Neumann and Dirichlet boundary conditions at that end. If one string end is subject to a Dirichlet boundary condition, say $g_1 = 1/2$, the spectrum is discrete, with eigenvalues $(m + 1/2 \pm g_2)^2$, and if both ends have Neumann boundary conditions, $g_1 = g_2 = 0$, the spectrum varies continuously over all positive real numbers. For all other values of g_1 and g_2 the spectrum exhibits band structure. It starts with a gap from zero energy to $(g_1 - g_2)^2/4$, followed by a band of width $(\frac{1}{2} - g_1)(\frac{1}{2} - g_2)$. After that two types of gaps alternate. For each positive integer m, there are gaps of width $2m(g_1 - g_2)$ and $(2m+1)(g_1 + g_2)$ in λ, whereas the intervening bands have widths $(\frac{1}{2} - g_1)(2m - (\frac{1}{2} - g_2))$ and $(\frac{1}{2} - g_1)(2m + (\frac{1}{2} - g_2))$. The limit $g_1 = g_2$ is very special because then half of all the gaps close.

At finite boson radius the general band structure (4.7) associated with boundary condition changing fields is rendered discrete by the restriction to allowed momenta. At the self-dual radius the result is a particularly simple set of eigenvalues $(n \pm g_-)^2$, with $n \in \mathbb{Z}$.

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4.3 Open string spectrum in the half-brane theory

By setting $\bar{g}_1 = \bar{g}_2 = 0$ in equation (4.5) we can obtain the open string spectrum for the 'half-brane' theory (1.3)

$$S = \frac{1}{4\pi}\int d^2z\, \partial\Phi\bar{\partial}\Phi - \frac{1}{2}g_0 \int d\tau e^{i\Phi(\tau)/\sqrt{2}}\,. \tag{4.8}$$

The trace in (4.6) reduces to $2\cos(2\pi k)$ and we immediately find that $\lambda = k$. The spectrum is therefore unaffected by the half-brane boundary interaction. This result was previously found in [15] by a different method.

5. Boundary correlation functions

In section 3.2 we argued that correlation functions involving discrete boundary fields are not affected by the boundary interaction. This only applies for discrete fields and correlators of more general boundary fields will depend on the boundary coupling in a non-trivial way.

5.1 Boundary fields carrying generic momentum

We can anticipate the structure of low-order boundary amplitudes from conformal symmetry. The boundary primary field at generic momentum is $V_{n,p}(x)$, where $n \geq 0$ labels the band that the corresponding open string state lies in and p is a momentum in the first Brillouin zone, $-\frac{1}{\sqrt{2}} \leq p \leq \frac{1}{\sqrt{2}}$. For two- and three-point functions one finds

$$\begin{aligned}
\langle V_i(x_i)V_j(x_j)\rangle &= \frac{G_{ij}}{|x_i - x_j|^{2\Delta_i}}\,, \\
\langle V_i(x_i)V_j(x_j)V_k(x_k)\rangle &= \frac{C_{ijk}}{|x_i - x_j|^{\Delta_i+\Delta_j-\Delta_k}|x_j - x_k|^{\Delta_j+\Delta_k-\Delta_i}|x_k - x_i|^{\Delta_k+\Delta_i-\Delta_j}}\,,
\end{aligned} \tag{5.1}$$

where V_i is short for V_{n_i,p_i} and $\Delta_i = (n_i + \lambda(p_i))^2$ is the scaling dimension obtained from (4.4), or (4.7) in the case of boundary condition changing fields. It remains to determine the g-dependence of the coefficients G_{ij} and C_{ijk} in the interacting theory. Conformal invariance requires the boundary scaling dimensions of V_i and V_j in the boundary two-point function to be equal. We can normalize the boundary fields in such a way that $G_{ij} = \delta_{n_i,n_j}$ if $p_i + p_j = 0$ and $G_{ij} = 0$ otherwise. This means that, once we have determined the open string spectrum, we know the two-point functions of all boundary fields in the interacting theory. This way the coupling dependence of G_{ij} is shifted into the bulk-to-boundary OPE coefficients for fields at generic momentum. These OPE coefficients can in principle be obtained from bulk two-point functions of the form $\langle e^{iq(\phi(z)+\bar{\phi}(\bar{z}))}e^{iq'(\phi(w)+\bar{\phi}(\bar{w}))}\rangle$ in the limit where the bulk operators approach the boundary, but here $q, q' \in \mathbb{R}$ are generic momenta, so these bulk two-point functions can not be computed by the method of rotated images. Similarly, the boundary OPE coefficients C_{ijk} for general momenta will in general depend on g in a way that cannot be determined by our SU(2) methods.

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5.2 Mixed bulk-boundary correlation functions

It is also of interest to compute mixed correlation functions involving both bulk and boundary fields. At generic momentum the best we can do is to once again use conformal invariance to constrain low order amplitudes. For a two-point function involving a boundary field at x and a bulk field at z the dependence on x, z and $\bar{z}$ is completely determined by the boundary and bulk scaling dimensions of the respective fields. We have derived how the boundary scaling dimensions depend on the coupling g but the two-point function also contains the relevant bulk-to-boundary OPE coefficient, which involves g in a way that we have not determined.

Mixed correlation functions that involve only discrete fields, both in the bulk and on the boundary, can be computed by treating the discrete boundary fields as limits of purely holomorphic bulk discrete fields, as described in section 3.2,

$$\Psi_{JM}(x) = \lim_{w \to x} \psi_{JM}(w)\psi_{00}(\bar{w}) \,, \tag{5.2}$$

and using the method of rotated images. Alternatively, we can define the boundary field $\Psi_{JM}(x)$ as the limit obtained when a purely anti-holomorphic bulk discrete field approaches the boundary,

$$\hat{\Psi}_{JM}(x) = \lim_{\bar{w} \to x} \psi_{00}(w)\psi_{JM}(\bar{w}) \,. \tag{5.3}$$

In the free theory these two definitions lead to identical results but this is no longer the case in the interacting theory. This is illustrated by two-point functions involving one boundary field and one in the bulk. Consider for example $\langle \Psi_{10,11}(z,\bar{z})\Psi_{11}(x)\rangle$, which vanishes by momentum conservation in the free theory. If we use (5.2) and implement the method of rotated images by moving the integration contours of the boundary currents into the lower half-plane we find

$$\begin{aligned} \langle \Psi_{10,11}(z,\bar{z})\Psi_{11}(x)\rangle_g &= \lim_{w \to x} \sum_{\bar{m}=-1}^{1} \mathcal{D}^1_{1,\bar{m}}(g)\, \langle \psi_{10}(z)\psi_{1\bar{m}}(z^*)\psi_{11}(w)\rangle \\ &= \frac{\sqrt{2}\sin^2 \pi g}{(z-z^*)|z-x|^2} \,. \end{aligned} \tag{5.4}$$

It is easily checked that the end result is the same when the integration contours are moved into the upper half-plane. If we instead choose (5.3) as our boundary field the two-point function vanishes, as is easily seen by moving the boundary integration contours into the upper half-plane,

$$\begin{aligned} \langle \Psi_{10,11}(z,\bar{z})\hat{\Psi}_{11}(x)\rangle_g &= \lim_{w^* \to x} \sum_{m=-1}^{1} \mathcal{D}^1_{0,m}(-g)\, \langle \psi_{1m}(z)\psi_{11}(z^*)\psi_{11}(w^*)\rangle \\ &= 0 \,. \end{aligned} \tag{5.5}$$

Each term in the sum is zero by momentum conservation in the free chiral theory.

The difference between (5.4) and (5.5) is not a sign of any pathology in the theory but rather serves as a reminder that the two definitions (5.2) and (5.3) select different bases for the discrete boundary operators, which are related to one another by a global

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SU(2) rotation. Boundary amplitudes of discrete operators are independent of the choice of basis but the bulk-to-boundary OPE coefficients are not and this is reflected in mixed amplitudes involving bulk and boundary discrete fields.

6. Discussion

In this paper we set out to compute correlation functions of primary fields in the two-dimensional boundary conformal field theory of a scalar field interacting with a critical periodic boundary potential. By extending the method of rotated images introduced in [11] we have shown how to calculate general correlation functions of both bulk and boundary primary fields when the boson is compactified at the self-dual radius. At other allowed boson radii our methods still generate all correlation functions of the discrete fields, which carry integer multiples of the lattice momentum defined by the periodic interaction, but not for fields with fractional momenta compared to the lattice momentum. In particular, we would like to compute correlation functions for primary fields carrying continuum values of the momentum in the theory at infinte boson radius.

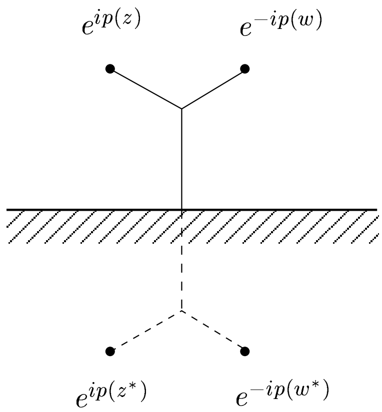

Figure 6: Method of rotated images applied in the crossed channel for bulk operators with opposite momenta.

It should in principle be possible to use the method of rotated images to calculate bulk two-point functions of operators with generic momenta. These are expected to be non-vanishing whenever the two momenta add up to a lattice momentum, but let us focus on the simplest case of opposite momenta. One would start by reflecting the anti-holomorphic fields in the two-point function

$$\left\langle e^{ip\phi(z)} e^{ip\bar{\phi}(\bar{z})} e^{-ip\phi(w)} e^{-ip\bar{\phi}(\bar{w})} \right\rangle, \tag{6.1}$$

through the real axis and then apply the OPE of the free chiral theory to the holomorphic image fields in the lower half-plane. The fields on the right-hand side of this OPE all carry momentum zero and can therefore be written as linear combinations of discrete primary fields with $m = 0$ and descendants of these primaries. The method of rotated images is then applied term by term, rotating descendants in the same representation as the corresponding primaries. One then also writes out the OPE of the two holomorphic fields in (6.1) and finally carries out the sum over the remaining chiral two point functions. What makes this approach challenging is the non-trivial decomposition of operators of the type $\partial^{n_1}\phi \cdots \partial^{n_k}\phi$ into different J-families. We have calculated the first few terms but in order to make use of this method one really needs to find the exact sum of the series, or in other words work out the exact conformal block that encodes the SU(2) rotation on the image fields. It would be interesting to see how the g-dependence of the scaling dimension of boundary operators arises in such an approach.

Acknowledgments

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A. Rotation coefficients

In this appendix we quote a general formula [24] for the rotation coefficients $\mathcal{D}^j_{m,n}(g)$ in (2.16) and give explicit examples for low values of j. For simplicity we take $g=\bar{g}$. In this case the rotation coefficients can all be expressed in terms of $\sin\pi g$ and $\cos\pi g$ as follows

$$\mathcal{D}^j_{m,n}(g)=\sum_\ell \frac{\sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}}{(j-m-\ell)!(j+n-\ell)!\ell!(m-n+\ell)!}(\cos\pi g)^{2j-2\ell+n-m}(i\sin\pi g)^{2\ell+m-n}\,, \quad (A.1)$$

where the sum is over integers ℓ and the range of summation is cut off when the argument in any one of the factorials in the denominator goes negative. When $g=\bar{g}$ the rotation coefficients are symmetric, $\mathcal{D}^j_{m,n}(g)=\mathcal{D}^j_{n,m}(g)$.

The representation for $j=0$ is trivial, $\mathcal{D}^0_{0,0}=1$. In the $j=1/2$ representation we have

$$\mathcal{D}^{1/2}(g)=\begin{pmatrix}\cos\pi g & i\sin\pi g\\ i\sin\pi g & \cos\pi g\end{pmatrix}\,, \quad (A.2)$$

and for $j=1$ we find

$$\mathcal{D}^1(g)=\begin{pmatrix}\cos^2\pi g & \frac{i}{\sqrt{2}}\sin 2\pi g & -\sin^2\pi g\\ \frac{i}{\sqrt{2}}\sin 2\pi g & \cos 2\pi g & \frac{i}{\sqrt{2}}\sin 2\pi g\\ -\sin^2\pi g & \frac{i}{\sqrt{2}}\sin 2\pi g & \cos^2\pi g\end{pmatrix}\,. \quad (A.3)$$

The extension to higher values of j is straightforward.

The rotation coefficients satisfy a number of useful relations including

$$\begin{aligned}\mathcal{D}^j_{m,n}(-g) &= (-1)^{n-m}\,\mathcal{D}^j_{-n,-m}(g)\,,\\ \mathcal{D}^{j_1}_{m_1,n_1}(g)\mathcal{D}^{j_2}_{m_2,n_2}(g) &= \sum_{J=|j_1-j_2|}^{j_1+j_2} C^{J(m_1+m_2)}_{j_1m_1,j_2m_2}C^{J(n_1+n_2)}_{j_1n_1,j_2n_2}\mathcal{D}^J_{(m_1+m_2),(n_1+n_2)}(g)\,,\end{aligned} \quad (A.4)$$

where $C^{JM}_{j_1m_1,j_2m_2}$ are Clebsch-Gordan coefficients.

B. Open string spectrum from closed string channel

In this appendix we give an alternative derivation of the spectrum (4.7) of open strings with endpoints interacting with two different boundary potentials. The calculation is a straightforward generalization of that presented in the "Note added" at the end of [11] and, for the most part, we use their notation.

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We consider a non-compact boson on a strip of width ℓ with different boundary potential strengths $g_1, g_2 \in \mathbb{R}$ at each boundary. The partition function in the closed string channel is

$$Z = \langle B, g_1 | q^{L_0 + \tilde{L}_0} | B, g_2 \rangle \,, \tag{B.1}$$

where $q = e^{-2\pi\ell/\beta}$ with β the parameter length of the closed string, and $|B, g\rangle$ is the boundary state induced by the boundary interaction (1.7). The exact expression for this boundary state, obtained in [11], is given by[6]

$$|B, g\rangle = \frac{1}{\sqrt{2\pi}} \sum_{j=0,\frac{1}{2},1\ldots} \sum_{m=-j}^{j} \mathcal{D}^{j}_{m,-m}(g)\, |j, m, m\rangle\rangle \,, \tag{B.2}$$

where $|j, m, m'\rangle\rangle$ denotes the reparametrization invariant Ishibashi state [29] based on the bulk discrete primary field $\Psi_{j,m;j,m'}(z, \bar{z})$.

The partition function is thus given by

$$Z = \frac{1}{2\pi} \sum_{j_1,m_1} \sum_{j_2,m_2} \mathcal{D}^{j_1}_{m_1,-m_1}(g_1)^* \, \mathcal{D}^{j_2}_{m_2,-m_2}(g_2) \langle\langle j_1, m_1, m_1 | q^{L_0+\tilde{L}_0} | j_2, m_2, m_2 \rangle\rangle . \tag{B.3}$$

The sum over descendants in the Ishibashi states produces a Virasoro character,

$$Z = \frac{1}{2\pi} \sum_{j=0,\frac{1}{2},1,\ldots} \chi^{\mathrm{Vir}}_j(q^2) \sum_{m=-j}^{j} \mathcal{D}^{j}_{m,-m}(g_1)^* \, \mathcal{D}^{j}_{m,-m}(g_2) \,, \tag{B.4}$$

leaving us with a sum over rotation coefficients that can be converted to an integral over SU(2) characters as follows,

$$\begin{aligned} \sum_{m=-j}^{j} \mathcal{D}^{j}_{m,-m}(g_1)^* \, \mathcal{D}^{j}_{m,-m}(g_2) &= \sum_{m=-j}^{j} \langle j, m | e^{2\pi i g_2 J_1} | j, -m \rangle \langle j, -m | e^{-2\pi i g_1 J_1} | j, m \rangle \\ &= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \sum_{m=-j}^{j} \langle j, m | e^{i\phi J_3} e^{2\pi i g_2 J_1} e^{-i\phi J_3} e^{-2\pi i g_1 J_1} | j, m \rangle \\ &= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \frac{\sin(2j+1)\alpha/2}{\sin\alpha/2} \,, \\ &= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \chi^{\mathrm{SU}(2)}_j(\alpha) \,, \end{aligned} \tag{B.5}$$

where the net rotation angle α can, for example, be determined using the $j = 1/2$ representation of SU(2) for which $J_i = \frac{1}{2}\sigma_i$, with σ_i Pauli matrices. After some straightforward algebra one finds

$$\sin^2\left(\frac{\alpha}{4}\right) = \sin^2(\pi g_-) \cos^2\left(\frac{\phi}{2}\right) + \cos^2(\pi g_+) \sin^2\left(\frac{\phi}{2}\right) , \tag{B.6}$$

[6] Our normalization convention for $|B, g\rangle$ differs from the one adopted in [11].

where $g_\pm = \frac{1}{2}(g_1 \pm g_2)$. This reduces to (4.7) if we make the identifications $\alpha = 4\pi\lambda$ and $\phi = 2\pi k$. Finally, we can rewrite the partition function Z in the form (4.3) by inserting (B.5) into (B.4) and then converting to the open string channel by the following modular transformation

$$\sum_{j=0,\frac{1}{2},1,\ldots} \frac{1}{\sqrt{2}} \chi_j^{\mathrm{SU}(2)}(\alpha)\, \chi_j^{\mathrm{Vir}}(q^2) = \frac{1}{\eta(\omega)} \sum_{m=-\infty}^{\infty} \omega^{(m+\frac{\alpha}{4\pi})^2} . \tag{B.7}$$

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